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Solitons of Nonlinear Scalar Electrodynamics in General Relativity

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Solitons with spherical and/or cylindrical symmetry in the interacting system of scalar, electromagnetic, and gravitational fields have been obtained. As a particular case it is shown that the equations of motion admit a special kind of solution with a sharp boundary, known as droplets. For these solutions, the physical fields vanish and the space-time is flat outside of the critical sphere or cylinder. Therefore, the mass and the electric charge of these configurations are zero.

1. INTRODUCTION

Since the early history of elementary particle physics, attempts to construct a divergence-free theory have been undertaken. Mie (1912a,b) proposed a nonlinear modification of the Maxwell equations, with the nonlinear electric current of the form $j_\mu = (A_\nu A^\nu)^2 A_\mu$. Within the scope of this modification there exist regular solutions approximating the electron structure.

Rosen (1939) considered a system of interacting electromagnetic and complex scalar fields that also admitted the existence of localized particle-like solutions. Nevertheless, these two models suffered the same defect: the mass of the localized object turned out to be negative. Recently it was shown that this defect of nonlinear electrodynamics can be corrected within the framework of general relativity (Chugunov *et al.*, 1996).

The aim of this paper is to consider a self-consistent system of fields to obtain particle-like configurations in the framework of general relativity. We show that in the case of an electromagnetic, scalar, and gravitational field

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system with specific type of interactions there exist droplet-like solutions having zero electric charge and mass. It is noteworthy that the effective potentials in this case possess the confining property, i.e., create a strong repulsion on certain surfaces in configuration space.

2. FUNDAMENTAL EQUATIONS

As is known, there do not exist regular static spherically or cylindrically symmetric configurations within the framework of gauge-invariant nonlinear electrodynamics (Bronnikov and Shikin, 1985). One possible way to overcome this difficulty is the nonlinear generalization of electrodynamics, with the use of a Lagrangian explicitly containing the 4-potential A_μ, $\mu = 0, 1, 2, 3$, thus breaking the gauge invariance inside a small critical sphere or cylinder. The introduction of terms depending explicitly on potentials in the electromagnetic equations presents the possibility to give an alternative explanation of such phenomena as inelastic photon–photon interactions (Novello and Salim, 1979), galactic redshift anomalies (Schiff, 1969; Peckev *et al.*, 1972; Goldhaber and Nieto, 1971), electric screening at low temperature in the limit of indirect interaction of photons with the thermal neutrino background (Woloshyn, 1983), the excess of high-energy photons in the isotropic flux (Ljubicic *et al.*, 1979), avoidance of the Big Bang singularity (Novello and Heintzmann, 1983), and the origin of self-focused beam in the effective nonlinear vector field theory (Bisshop, 1972). The corresponding terms appear in our scheme due to the interaction between the electromagnetic and scalar fields. This interaction being negligible at large distances, the Maxwellian structure of the electromagnetic equations (and therefore the gauge invariance) is reinstated far from the center of the system.

We choose the Lagrangian in the form (Bronnikov and Shikin, 1985)

$$L = \frac{R}{2\kappa} - \frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{8\pi} \varphi_{,\alpha}\varphi^{,\alpha}\Psi(I) \tag{2.1}$$

where $\kappa = 8\pi G$ is the Einstein gravitational constant and the function $\Psi(I)$ of the invariant $I = A_\mu A^\mu$ characterizes the interaction between the scalar φ and electromagnetic A_μ fields. In the sequel the function $\Psi(I)$ will be viewed as arbitrary; thus the Lagrangian (2.1) defines the class of models parametrized by $\Psi(I)$. Schwinger (1951) used a special method to compute the effective coupling between a zero-spin neutral meson and the electromagnetic field using some functions of the electromagnetic field. Thus our approach to generate an effective Lagrangian generalizes the one proposed by Schwinger.

The particular choice of $\Psi(I)$ will be made to obtain droplet-like configurations. The field equations corresponding to the Lagrangian (2.1) read

$$\mathcal{G}_\mu^\nu = -\kappa T_\mu^\nu \tag{2.2}$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{\alpha\beta} \varphi_{,\beta} \Psi) = 0 \tag{2.3}$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} (\sqrt{-g} F^{\alpha\beta}) - (\varphi_{,\beta} \varphi^{,\beta}) \Psi_I A^\alpha = 0 \tag{2.4}$$

where $\Psi_I = d\Psi/dI$ and $\mathcal{G}_\mu^\nu = R_\mu^\nu - \delta_\mu^\nu R/2$ is the Einstein tensor. One can write the energy-momentum tensor of the interacting matter fields in the form

$$T_\mu^\nu = (1/4\pi)[\varphi_{,\mu}\varphi^{,\nu}\Psi(I) - F_{\mu\alpha}F^{\nu\alpha} + \varphi_{,\alpha}\varphi^{,\alpha}\Psi_I A_\mu A^\nu]$$
$$- \delta_\mu^\nu \left[\frac{1}{8\pi} \varphi_{,\beta}\varphi^{,\beta}\Psi(I) - \frac{1}{16\pi} F_{\alpha\beta}F^{\alpha\beta} \right] \tag{2.5}$$

3. CONFIGURATIONS WITH SPHERICAL SYMMETRY

Searching for the static, spherically symmetric solutions to the system of equations (2.2)–(2.4), we consider the metric in the form (Bronnikov and Kovalchuk, 1980)

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} d\xi^2 - e^{2\beta}[d\theta^2 + \sin^2\theta \, d\phi^2] \tag{3.1}$$

with ξ being the radial variable. Let us now formulate the requirements to be fulfilled by particle-like solutions (PLS). These are (Bronnikov *et al.*, 1993):

(a) Stationarity [applied to the metric (3.1)], i.e.,

$$\alpha = \alpha(\xi), \qquad \beta = \beta(\xi), \qquad \gamma = \gamma(\xi)$$

(b) Regularity of the metric and the matter fields in the whole space-time.

(c) Asymptotically Schwarzschild metric and corresponding behavior of the field functions.

In view of requirement (a), it is convenient to choose the harmonic ξ coordinate ($\Box\xi = 0$) in (3.1) to satisfy the subsidiary condition (Bronnikov *et al.*, 1979)

$$\alpha = 2\beta + \gamma \tag{3.2}$$

The corresponding coordinate in flat space-time is just $\xi = 1/r$. With the constraint (3.2) the system of Einstein equations (2.2) reads

$$e^{-2\alpha}(2\beta'' - U) - e^{-2\beta} = -\kappa T_0^0 \tag{3.3}$$

$$e^{-2\alpha} U - e^{-2\beta} = -\kappa T_1^1 \tag{3.4}$$

$$e^{-2\alpha}(\beta'' + \gamma'' - U) = -\kappa T_2^2 = -\kappa T_3^3 \tag{3.5}$$

where $U = \beta'^2 + 2\beta'\gamma'$, and prime denotes differentiation with respect to ξ Note that the field functions, as well as the components of the metric tensor, depend on the single spatial variable ξ. Assuming the electromagnetic field to be determined by the time component $A_0 = A(\xi)$ of the 4-potential, one finds the unique nontrivial component of the field tensor $F_{10} = A'$, and the invariant I reduces to $I = e^{-2\gamma}A^2(\xi)$.

One can write the nonzero components of the energy-momentum tensor (2.5) as follows:

$$T_0^0 = (1/8\pi)e^{-2\alpha}[A'^2e^{-2\gamma} + \varphi'^2(\Psi - 2A^2e^{-2\gamma}\Psi_I)] \tag{3.6}$$

$$T_1^1 = -T_2^2 = -T_3^3 = (1/8\pi)e^{-2\alpha}[A'^2e^{-2\gamma} - \varphi'^2\Psi] \tag{3.7}$$

Adding equations (3.4) and (3.5) and using the property $T_1^1 + T_2^2 = 0$, one obtains the differential equation

$$\beta'' + \gamma'' - e^{2(\beta+\gamma)} = 0$$

with the solution (Bronnikov, 1973)

$$e^{-(\beta+\gamma)} = \mathcal{S}(k, \xi) = \begin{cases} k^{-1}\,\mathrm{sh}\, k\xi, & k > 0 \\ \xi, & k = 0 \\ k^{-1}\sin k\xi, & k < 0 \end{cases} \tag{3.8}$$

depending on the constant k. Notice that another constant of integration is trivial, so that $\xi = 0$ corresponds to the spatial infinity, where $e^\gamma = 1$ and $e^\beta = \infty$. Without loss of generality one can choose $\xi > 0$.

The scalar field equation (2.3) has the evident solution

$$\varphi' = CP(I) \tag{3.9}$$

where $P(I) = 1/\Psi(I)$ and C is the integration constant. Putting (3.9) into (2.4), one gets the equation for the electromagnetic field

$$(e^{-2\gamma}A')' - C^2P_Ie^{-2\gamma}A = 0 \tag{3.10}$$

where the second term could be naturally interpreted as the induced nonlinearity. In view of (3.9), one rewrites the Einstein equation (3.4) and the result of adding the equations (3.3) and (3.4) as follows:

$$\gamma'^2 = -G(C^2P - A'^2e^{-2\gamma}) + K, \qquad K = k^2\,\mathrm{sign}\,k \tag{3.11}$$

$$\gamma'' = Ge^{-2\gamma}(A'^2 + C^2A^2P_I) \tag{3.12}$$

One can easily check that equation (3.11) is the first integral of equations (3.10) and (3.12). Eliminating the term (P_IA) between (3.10) and (3.12), one gets the equation

$$\gamma'' = G(AA'e^{-2\gamma})' \tag{3.13}$$

with the evident first integral

$$\gamma' = GAA'e^{-2\gamma} + C_1, \qquad C_1 = \text{const} \tag{3.14}$$

Let us consider the simple case $C_1 = 0$. Then from (3.14) we get

$$e^{2\gamma} = GA^2 + H, \qquad H = \text{const} \tag{3.15}$$

Substituting γ' and $e^{2\gamma}$ from (3.14) and (3.15) into (3.10), we find for $A(\xi)$ the differential equation

$$A'^2(GA^2 + H)^{-2} = (GC^2P - K)/GH \tag{3.16}$$

which can be solved by quadrature:

$$\int \frac{dA}{(GA^2 + H)\sqrt{GC^2P - K}} = \pm(1/\sqrt{GH})(\xi - \xi_0), \qquad \xi_0 = \text{const} \tag{3.17}$$

It is clear that the configuration obtained has a center if and only if $e^{\beta} = 0$ at some $\xi = \xi_c$. One can show (Bronnikov *et al.*, 1979) that the conditions for the center $\xi_c = \infty$ to be regular imply $K = 0$ and the following behavior of the field quantities in the vicinity of the point $\xi_c = \infty$:

$$\gamma' = O(\xi^{-2}), \qquad A \to A_c \neq \infty, \qquad A' \to 0$$
$$\xi^4 P(I) \to 0, \qquad |\xi^4 I P_I| < \infty \tag{3.18}$$

In view of (3.18), we deduce from (3.14) that $C_1 = 0$ in accordance with the earlier supposition.

Now we can write the boundary conditions on the surface of the critical sphere $\xi = \xi_0$:

$$T_\mu^\nu = A = A' = 0, \qquad e^\gamma = 1, \qquad e^\beta = 1/\xi_0 > 0 \tag{3.19}$$

Due to (3.19) and (3.15), we infer that $H = 1$. The condition $K = 0$ leads to $k = 0$ in (3.8) and the space-time (3.1) that fulfills the regularity conditions (3.18) takes the form

$$ds^2 = (GA^2 + 1)\, dt^2 - \frac{1}{\xi^2(GA^2 + 1)}\left(\frac{d\xi^2}{\xi^2} + [d\theta^2 + \sin^2\theta\, d\phi^2]\right) \tag{3.20}$$

We can finally write A and φ as follows:

$$\int \frac{dA}{(GA^2 + 1)\sqrt{P}} = \pm C(\xi - \xi_0) \tag{3.21}$$

$$\varphi = C\int P\, d\xi = \int \sqrt{P}e^{-2\gamma}\, dA = \int \frac{\sqrt{P}\, dA}{GA^2 + 1} \tag{3.22}$$

Let us now calculate the matter-field energy density:

$$T_0^0 = (C^2/8\pi)e^{-2\alpha}[P(1 + e^{2\gamma}) + 2IP_I(I)] \tag{3.23}$$

One can readily derive from (3.23) the energy E_f of the matter fields:

$$\begin{aligned} E_f &= \int d^3x \sqrt{-^3g}T_0^0 \\ &= (C/2) \int_{A(\xi=0)}^{A(\xi\to\infty)} dA\, e^{-3\gamma}[\sqrt{P}(1 + e^{2\gamma}) + 4I(\sqrt{P})_I] \end{aligned} \tag{3.24}$$

Thus the equations for the scalar and electromagnetic fields are completely integrated. As one sees, to write the scalar (φ) and vector (A) functions as well as the energy density (T_0^0) and energy of the material fields (E_f) explicitly, one has to give $P(I)$ in explicit form. Here we will give a detailed analysis for some concrete forms of $P(I)$.

I. Let us consider $P(I)$ in the form

$$P(I) = P_0(\lambda I - N)^s R(\lambda I), \qquad 2 \le s \le 3 \tag{3.25}$$

where $R(\lambda I)$ is some arbitrary, continuous, positive-definite function, having nontrivial value at the center; λ is the coupling parameter; $N > 0$ is some dimensionless constant that is equal to the value of λI at the center. The other constant P_0 is defined from the condition $P = 1$ at spatial infinity, $\xi = 0$. For $R = \text{const}$ one gets the simplest form of $P(I)$ that leads to regular solutions. In this case the energy density is positive if $\lambda I \ge N$.

(a) Choosing $P(I)$ in the form

$$P(I) = P_0(\lambda I - N)^2 \tag{3.26}$$

we get

$$A(\xi) = \sqrt{\frac{N}{\lambda - GN}}\,\mathrm{cth}\,\Lambda(\xi + \xi_1) \tag{3.27}$$

where $\Lambda = \sqrt{C^2NP_0(\lambda - GN)}$, the integration constant ξ_1 is defined from $A(0) = m/q$, with m and q being the mass and the charge of the system, respectively. In this case we get

$$P_0 = (\lambda m^2/q^2 - N)^{-2}, \qquad \lambda m^2/q^2 > N$$

Inasmuch $\sqrt{\lambda}m/|q| > \sqrt{N}$, then $\delta = \sqrt{G}m/|q| > \sqrt{GN/\lambda} = \sigma$. Taking $\delta < 1$ and $\sigma < 1$, we get the inequality

$$0 < \sigma < \delta < 1$$

Now we can rewrite P_0 in the form

$$P_0 = \frac{G^2}{\lambda^2}(\delta^2 - \sigma^2)^{-2}$$

The metric function $e^{2\gamma}$, electric field, and the total energy of the material field system can be written as

$$e^{2\gamma} = GA^2 + 1 = \frac{C^2}{q^2}\left[\frac{\sigma^2}{1-\sigma^2}\operatorname{cth}^2\Lambda\,(\xi + \xi_1) + 1\right] \tag{3.28}$$

$$|\mathbf{E}| = (-F_{10}F^{10})^{1/2} = \frac{\Lambda\sqrt{N}}{\sqrt{\lambda(1-\sigma^2)}}\,\frac{\xi^2}{\operatorname{sh}^2\Lambda(\xi + \xi_1)} \tag{3.29}$$

$$\begin{aligned} E_f = \frac{q}{2\sqrt{G}}\Bigg[&\frac{\delta - \sigma}{\delta + \sigma}\,\frac{\delta + 2\sigma}{3} + \frac{4(\delta^2 + \delta\sigma + \sigma^2) - 3}{3(\delta + \sigma)} \\ &+ \frac{1 - \sigma^2}{2(\delta^2 - \sigma^2)}\ln\frac{(1+\delta)(1-\sigma)}{(1-\delta)(1+\sigma)}\Bigg] \end{aligned} \tag{3.30}$$

As one sees,

$$E_f|_{\delta\to\sigma} \to \frac{q\delta}{\sqrt{G}} = m, \qquad E_f|_{\delta\to 1} \to \infty$$

The infinite value of E_f can be interpreted as the physical reason for the existence of the limitation $\delta < 1$.

(b) Let us consider the case with $I_c = 0$, choosing

$$P(I) = \lambda I \tag{3.31}$$

At the spatial infinity, where $I = I_0 = m^2/q^2$, we have $P = 1$, which leads to $\lambda = q^2/m^2$, i.e., the coupling constant is connected with mass and charge. In this case we get

$$A(\xi) = \frac{1}{\sqrt{G}\operatorname{sh} m(\xi + \xi_1)} \tag{3.32}$$

where, as in the previous case, ξ_1 is defined from $A(0) = m/q$. The metric function $e^{2\gamma}$, electric field, and the total energy of the material field system can be written as

$$e^{2\gamma} = \frac{C^2}{q^2}\operatorname{cth}^2[mC(\xi + \xi_1)/q] \tag{3.33}$$

$$|\mathbf{E}| = \frac{mC^2}{q^2\sqrt{G}}\,\frac{\xi^2\operatorname{ch}[mC(\xi + \xi_1)/q]}{\operatorname{sh}^2[mC(\xi + \xi_1)/q]} \tag{3.34}$$

$$E_f = \frac{q}{4\sqrt{G}}\left[3\delta\,\frac{1}{\delta}\ln(1 - \delta^2)\right] \tag{3.35}$$

Thus one sees that

$$E_f|_{\delta<<1} \approx m, \qquad E_f|_{\delta\to 1} \to \infty$$

II. A specific type of solution to the nonlinear field equations in flat space-time was obtained in a series of interesting articles (Werle, 1977, 1980, 1981, 1988). These solutions are known as droplet-like solutions or simply droplets. The distinguishing property of these solutions is the availability of some sharp boundary defining the space domain in which the material field happens to be located, i.e., the field is zero beyond this area. It was found that the solutions mentioned exist in field theory with specific interactions that can be considered as effective, generated by initial interactions of unknown origin. In contrast to the widely known soliton-like solutions, with field functions and energy density asymptotically tending to zero at spatial infinity, the solutions in question vanish at a finite distance from the center of the system (in the case of spherical symmetry) or from the axis (in the case of cylindrical symmetry). Thus, there exists a sphere or cylinder with critical radius r_0 outside of which the fields disappear. Therefore the field configurations have a droplet-like structure (Werle, 1977; Bronnikov *et al.*, 1991; Rybakov *et al.*, 1994a).

Let us now choose the function $P(I)$ as follows (Rybakov *et al.*, 1994b) (see Fig. 2):

$$P(J) = J^{(1-2/\sigma)}[(1 - J)^{1/\sigma} - J^{1/\sigma}]^2(1 - J) \tag{3.36}$$

where $J = GI$, $\sigma = 2n + 1$, $n = 1, 2, 3, \ldots$. Then on account of $K = 0$ and $H = 1$ we get from (3.17) the following expression for $A(\xi)$ (see Fig. 1):

$$A(\xi \le \xi_0) = 0, \qquad A(\xi \ge \xi_0) = (1/\sqrt{G})\left\{1 - \exp\left[-\frac{2C\sqrt{G}}{\sigma}(\xi - \xi_0)\right]\right\}^{\sigma/2} \tag{3.37}$$

As one can see from (3.37), the conditions (3.18) for the center to be regular and the matching conditions (3.19) on the surface of the critical sphere are fulfilled if $\sigma > 2$. It is also obvious from (3.37) that for $\xi < \xi_0$ the value of the square bracket turns out to be negative and $A(\xi)$ becomes imaginary, since σ is an odd number. Since we are interested in real $A(\xi)$ only, without loss of generality we may assume the value of $A(\xi)$ to be zero for $\xi \le \xi_0$, the matching at $\xi = \xi_0$ being smooth.

Recalling that $J = GA^2/(GA^2 + 1)$, we get from (3.37) that $J(\infty) = 1/2$ and $J(\xi_0) = 0$, thus implying

$$P(I)|_{\xi=\infty} = P(I)|_{\xi=\xi_0} = 0 \tag{3.38}$$

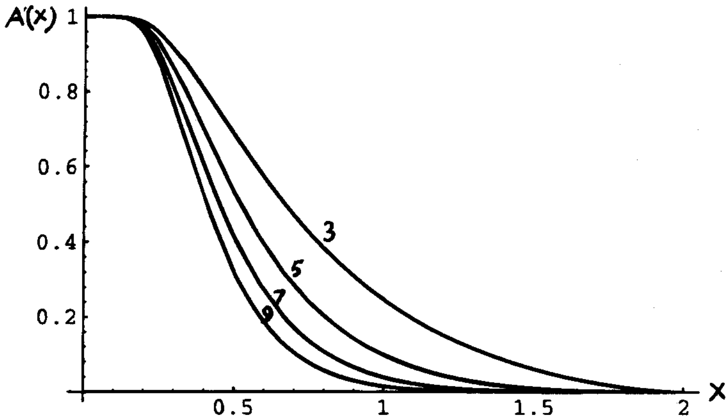

Fig. 1. Perspective view of droplet-like solution. The configurations are plotted for $\lambda = 1$, $x_0 = 2$, and $\sigma = 3, 5, 7, 9$.

This means that at $\xi = \xi_c = \infty$ and $\xi = \xi_0$, the interaction function $\Psi(I) = 1/P(I)$ is singular. It turns out nevertheless that the energy density T_0^0 is regular at these points due to the fact that it contains $\Psi(I)$ as a multiplier in the form

$$e^{-2\alpha}\varphi'^2\Psi = C^2 e^{-2\alpha} P(I) \tag{3.39}$$

which tends to zero as $\xi \to \xi_c$ or $\xi \to \xi_0$. As follows from (3.37), for the limiting case $\xi_0 = 0$, when the critical sphere goes to the spatial infinity and the solution in question is defined at $0 \leq \xi \leq \infty$, it appears that at spatial

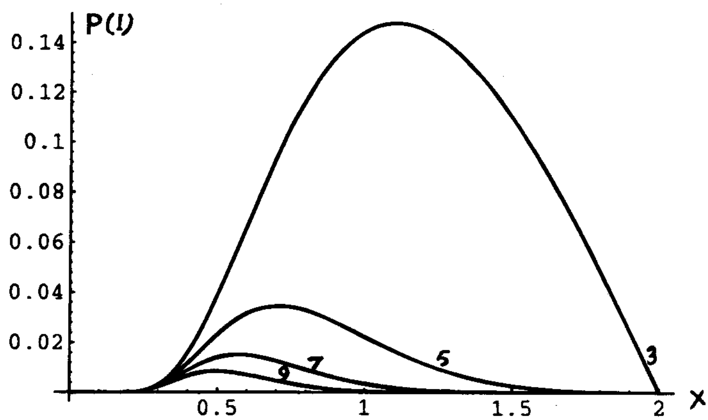

Fig. 2. Perspective view of the inverse function to the interaction one [i.e., $P(I)$] that provides us with the droplet-like configurations (Fig. 1). As is seen from Fig. 1, the stronger the interaction, the more localized is the corresponding droplet-like configuration.

infinity ($\xi = 0$) $A = 0$ and $P(I) = 0$. In this case we obtain the usual soliton-like configuration not possessing any sharp boundary.

It should be emphasized that at spatial infinity ($\xi = 0$) one can compare the metric found with the Schwarzschild one and the electric field with the Coulomb one, thus determining the total mass m and the charge q of the system:

$$Gm = -\gamma'(0), \qquad q = -A'(0)$$

Taking into account that $e^{2\gamma} = GA^2 + 1$, one can find through the use of (3.37) that for $\xi_0 = 0$, $A'(0) = -q = 0$ and $\gamma'(0) = -Gm = 0$. Therefore, the total energy of the soliton-like system, defined as the sum of the material field energy and that of the gravitational field, vanishes. If now one chooses the integration constant $\xi_0 > 0$, then the field configuration with the sharp boundary (droplet) appears. In this case for $\xi \leq \xi_0$ one obtains $A(\xi) = 0$ and $e^{2\gamma} = 1$, i.e., outside of the droplet the gravitational and electromagnetic fields disappear, which implies the vanishing of the total mass and the charge of the system. This unusual property makes the droplet-like object poorly visible for the outer observer.

It should be emphasized that the total energy is localized in the region ($\xi_0 \leq \xi < \infty$),

$$T_0^0(\xi)|_{\xi \to \infty} \to 0, \qquad T_0^0(\xi)|_{\xi \to \xi_0} \to 0 \tag{3.40}$$

namely, inside the critical sphere with the radius

$$R = \int_0^\infty d\xi\, e^{\alpha(\xi)} = \int_0^\infty d\xi/\xi^2\{[1 - e^{-2C\sqrt{G}(\xi-\xi_0)/\sigma}]^\sigma + 1\}^{1/2} < \infty$$

Taking into account that $e^{2\gamma} = 1/(1 - J)$ and $e^{-3\gamma}dA = dJ/2\sqrt{GJ}$, we rewrite the total energy of the material fields in terms of J:

$$E_f = \frac{C}{4\sqrt{G}} \int_0^{1/2} \left(4\frac{d\sqrt{JP}}{dJ} + \frac{\sqrt{PJ}}{1 - J}\right) dJ$$

The contribution of the first term of this equality is trivial for the choice of $P(I)$ in the form (3.36), as in this case $P(I)|_0 = P(I)|_{\lambda/2} = 0$. As $P(I)$ is positive and J lies in the interval (0, 1/2), one estimates

$$E_f = \frac{C}{4\sqrt{G}} \int_0^{1/2} \frac{\sqrt{PJ}}{1 - J}\, dJ > 0$$

Note that we consider the constant C to be positive. Since we know that the total energy of the droplet-like object is zero, this inequality implies the negativity of its gravitational energy. Thus the droplet-like configuration of the fields obtained is totally regular with zero total energy (including the energy of the proper gravitational field) and null electric charge and remains unobservable to one located outside the sphere with radius R (Rybakov *et al.*, 1992 1994b). In order to clarify the fact that the role of the gravitational field in forming the droplet-like configuration is not decisive, it is worthwhile to compare the solution obtained with that in the flat space-time, described by the interval

$$ds^2 = dt^2 - dr^2 - r^2[d\theta^2 + \sin^2\theta\, d\phi^2]$$

In the latter case equation (2.3) admits the solution

$$\varphi'(r) = -CP(I)/r^2 \tag{3.41}$$

Substituting (3.41) into (2.4), one finds that the equation for the electromagnetic field can be solved by quadrature:

$$\int \frac{dA}{\sqrt{P}} = \pm C\left(\frac{1}{r} - \frac{1}{r_0}\right), \qquad r_0 = \text{const} \tag{3.42}$$

Note that the droplet-like configuration $A(r)$ will be similar to (3.37) if one chooses a function $P(I)$ simpler than (3.36):

$$P(I) = J^{1-2/\sigma}(1 - J^{1/\sigma})^2, \qquad J = \lambda I \tag{3.43}$$

where $\lambda = \text{const}$, $\sigma = 2n + 1$, $n = 1, 2, 3, \ldots$. Then substituting (3.43) into (3.42), one gets the solution

$$A(r) = \frac{1}{\sqrt{\lambda}}\left\{1 - \exp\left[-\frac{2C\lambda}{\sigma}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]\right\}^{\sigma/2} \tag{3.44}$$

One can see from (3.44) that $A(r) = 0$ as $r \geq r_0$, i.e., the charge of the flat space-time droplet configuration also vanishes. For this solution the regularity conditions at the center $r = 0$ and on the surface of the critical sphere $r = r_0$ are evidently fulfilled. It similarly appears that for $r = \infty$ one finds the usual soliton-like structure with field vanishing as $r \to \infty$. The field energy E_f is defined as follows:

$$E_f = C\int_{A(r_0)}^{A(0)} dA\,(\sqrt{P} + IP_I/\sqrt{P}) = C\sqrt{P}I\,|_{A(r_0)}^{A(0)} \tag{3.45}$$

Considering that $PI = 0$ both at $r = 0$ and $r = r_0$, we arrive through (3.45) at $E_f = 0$. Thus in the flat space-time as well as for the self-gravitating system, the total energy and charge of the droplet-like configuration vanish.

4. CONFIGURATIONS WITH CYLINDRICAL SYMMETRY

Obviously, from the viewpoint of physics, the most interesting case is the spherically symmetric one; nevertheless, in some cases it is necessary to study the two-dimensional, cylindrically symmetric regular solutions in the vicinity of the symmetry axis [vortexes (Nielsen and Olesen, 1973), stringlike solutions (Terletsky, 1977)]. These solutions can describe realistic objects such as fluxions (Abrikosov, 1957) or light beams (Zakharov *et al.*, 1971) and can serve as the logical approximation to objects with toroidal structure (de Vega, 1978). Let us now search for static, cylindrically symmetric solutions to equations (2.2)–(2.4). In this case the metric can be chosen as follows (Bronnikov, 1979; Shikin, 1984):

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}dx^2 - e^{2\beta}\, d\phi^2 - e^{2\mu}\, dz^2 \tag{4.1}$$

The requirements to be fulfilled by soliton-like solutions in this case are (Shikin, 1995):

(a) Stationarity [applied to the metric (4.1)], i.e.,

$$\alpha = \alpha(x), \qquad \beta = \beta(x), \qquad \gamma = \gamma(x), \qquad \mu = \mu(x)$$

This means for (4.1) that all the components of the metric tensor depend on the single spatial coordinate $x \in [x_0, x_a]$, where x_a is the value of x on the axis of symmetry, defined by the condition $\exp[\beta(x_a)] = 0$, and x_0 is the value of x on the surface of the critical cylinder. The coordinates z and ϕ take their standard values: $z \in [-\infty, \infty]$, $\phi \in [0, 2\pi]$.

(b) Regularity of the metric and the matter fields in the whole space-time.

(c) Localization in space-time (with finite field energy)

$$E_f = \int T_0^0 \sqrt{-{}^3g}\, dV < \infty$$

Requirement (c) assumes the rapid decreasing of the energy density of the material field at spatial infinity, which together with (b) guarantees the finiteness of E_f. Let us note that E_f may be finite even for singular solutions on the axis. Requirement (b) means the regularity of material fields as well as the regularity of metric functions, which entails the demand of finiteness of the energy-momentum tensor of material fields all over the space. If the system considered contains scalar φ and electric $\mathbf{E}$ (or magnetic $\mathbf{H}$) fields, the regularity conditions on $x = x_a$ take the form (Bronnikov, 1979)

$$e^{\beta} = 0; \qquad |\gamma| < \infty; \qquad |\mu| < \infty; \qquad e^{2(\beta-\alpha)}(\beta')^2 = 1; \qquad e^{-2\alpha}(\gamma')^2 = 0$$
$$\{|\mathbf{E}| = 0; |\mathbf{H}_{\|}| < \infty; |\mathbf{H}_{\perp}| = 0\}; \qquad |T_\mu^\nu| < \infty \tag{4.2}$$

where $\mathbf{H}_{\|}$ and $\mathbf{H}_{\perp}$ are the longitudinal and transverse magnetic fields, respectively, defined as chronometric invariants (Mitskevich, 1969). In view of requirement (a), it is convenient to choose the coordinate x in (4.1) to satisfy the subsidiary condition (Shikin, 1984)

$$\alpha = \beta + \gamma + \mu$$

which enables us to present the system of the Einstein equations in the form

$$\mu'' + \beta'' - V = -\kappa T_0^0 \, e^{2\alpha} \tag{4.3}$$

$$\mu'\beta' + \beta'\gamma' + \gamma'\mu' = V = -\kappa T_1^1 \, e^{2\alpha} \tag{4.4}$$

$$\gamma'' + \beta'' - V = -\kappa T_2^2 \, e^{2\alpha} \tag{4.5}$$

$$\mu'' + \gamma'' - V = -\kappa T_3^3 \, e^{2\alpha} \tag{4.6}$$

As in the preceding section, the electromagnetic field is described by the time component of the 4-potential $A_0(x) = A(x)$ and by the component $F_{10} = dA/dx = A'$ of the field strength tensor, and the energy-momentum tensor of interacting fields is defined by equations (3.6), (3.7).

Adding equations (4.4) and (4.5) and using (3.7), one obtains the simple equation

$$\gamma'' + \beta'' = 0 \tag{4.7}$$

with the solution

$$\beta(x) + \gamma(x) = C_2 x, \qquad C_2 = \text{const} \tag{4.8}$$

Notice that the second integration constant in (4.8) can be taken as trivial, since it determines only the choice of scale.

In a similar way, the addition of equations (4.4) and (4.6) leads to the equation

$$\gamma'' + \mu'' = 0 \tag{4.9}$$

with the solution

$$\mu(x) + \gamma(x) = C_3 x, \qquad C_3 = \text{const} \tag{4.10}$$

whereas the subtraction of (4.5) and (4.6) gives

$$\beta'' - \mu'' = 0 \tag{4.11}$$

with the solution

$$\beta(x) - \mu(x) = C_4 x, \qquad C_4 = \text{const} \tag{4.12}$$

Solving equation (2.2) in the metric (4.1), one gets the same result as in (3.9), i.e.,

$$\varphi'(x) = CP(I) \tag{4.13}$$

Substituting (4.13) into (2.4), one finds an equation for the electromagnetic field identical to (3.10), i.e.,

$$(e^{-2\gamma}A')' - C^2P_I e^{-2\gamma}A = 0 \tag{4.14}$$

where the second term could be naturally interpreted as the induced nonlinearity. Now, as in the previous case, we use equation (4.4) and sum equations (4.3) and (4.4), which in view of (4.8) and (4.10), take the form

$$\gamma'^2 - C_2C_3 = -G(C^2P - A'^2e^{-2\gamma}) \tag{4.15}$$

$$\gamma'' = Ge^{-2\gamma}(A'^2 + C^2A^2P_I) \tag{4.16}$$

Elimination of P_IA between equations (4.14) and (4.16) gives the equation

$$\gamma'' = G(AA'e^{-2\gamma})' \tag{4.17}$$

with the evident first integral

$$\gamma' = GAA'e^{-2\gamma} + C_1, \qquad C_1 = \text{const} \tag{4.18}$$

Integrating (4.18) under the choice $C_1 = 0$, one again obtains

$$e^{2\gamma} = GA^2 + H, \qquad H = \text{const} \tag{4.19}$$

Finally, substituting γ' from (4.18) and $e^{2\gamma}$ from (4.19) into (4.15), one gets the equation for $A(x)$:

$$A'^2(GA^2 + H)^{-2} = (GC^2P - C_2C_3)/GH \tag{4.20}$$

Equation (4.20) can be solved by quadrature:

$$\int \frac{dA}{(GA^2 + H)\sqrt{GC^2P - C_2C_3}} = \pm \frac{1}{\sqrt{GH}}(x - x_0) \tag{4.21}$$

Let us formulate regularity conditions to be satisfied by the solutions to equations (2.2)–(2.4) on the axis of symmetry defined by the value $x = x_a$, where $\exp[\beta(x_a)] = 0$. Since according to the regularity conditions formulated earlier, $|\gamma(x_a)| < \infty$ and $|\beta(x_a)| < \infty$, from (4.8) and (4.12) one gets $\beta(x) \approx C_2x \to -\infty$ (whereas $x_a = -\infty$ if $C_2 > 0$ and $x_a = +\infty$ if $C_2 < 0$); $\beta(x) \approx C_4x \to -\infty$ (whereas $x_a = -\infty$ if $C_4 > 0$ and $x_a = +\infty$ if $C_4 < 0$). This leads to $C_2 = C_4$, $\gamma(x) \equiv -\mu(x)$, and $\alpha(x) \equiv \beta(x)$. As one sees, from $\gamma(x) \equiv -\mu(x)$ it follows that $C_3 = 0$. The regularity conditions are similar to (3.18) for the case of spherical symmetry, implying that the following relations hold as $x \to x_a = \infty$:

$$\gamma' \to 0, \qquad A \to A_c \neq \infty, \qquad A' \to 0$$

$$e^{2|C_2|x}P(I) \to 0, \qquad e^{2|C_2|x}|IP_I| < \infty \tag{4.22}$$

Boundary conditions on the surface of the critical cylinder $x = x_a$ can be written as follows:

$$T^{\nu}_{\mu} = A = A' = 0, \qquad e^{\gamma} = 1, \qquad e^{\beta} = e^{-|C_2|x} > 0 \tag{4.23}$$

The conditions (4.23) together with the relation $e^{2\gamma} = GA^2 + H$ imply that $H = 1$. Therefore the metric (4.1) that satisfies the regularity conditions reads:

$$ds^2 = (GA^2 + 1)\, dt^2 - \frac{1}{GA^2 + 1}\, [e^{2C_2x}(dx^2 + d\phi^2) + dz^2] \tag{4.24}$$

As in the previous case, we will study the system for different $P(I)$.

I. Note that some class of regular solutions can be obtained by choosing $P(I)$ in the form

$$P(I) = P_0(\lambda I - N)^s Q(\lambda I) \tag{4.25}$$

where $Q(\lambda I)$ is some arbitrary, continuous, positive-definite function having nontrivial value at the center; λ is the coupling parameter; $N > 0$ is some dimensionless constant that is equal to the value of λI at the center. The other constant, P_0, is defined from the condition $P = 1$ at spatial infinity, $x = \pm\infty$. For $Q(\lambda I) = \text{const}$ one gets the simplest form of $P(I)$ that leads to regular solitons. As in the spherically symmetric case, for the regular solutions, $\lambda \geq GN$.

(a) Choosing $P(I)$ in the form

$$P(I) = P_0(\lambda I - N)^2 \tag{4.26}$$

we get

$$A(x) = \sqrt{\frac{N}{\lambda - GN}}\, \text{th}\, bx \tag{4.27}$$

where $b = \sqrt{C^2 N P_0(\lambda - GN)}$; the integration constant x_1 is taken to be trivial. The regularity condition implies $b \geq 1$. The metric function $e^{2\gamma}$, radial electric field, and the total energy of the material field system can be written as

$$e^{2\gamma} = \frac{\lambda}{\lambda - GN}\left(1 - \frac{GN}{\lambda\, \text{ch}^2 bx}\right) \tag{4.28}$$

$$|\mathbf{E}| = |C| e^{\gamma-\beta}\sqrt{P(I)} \tag{4.29}$$

$$E_f = \frac{\lambda C}{2G\sqrt{G}}\left(\frac{\sigma}{\sqrt{1 - \sigma^2}} - \frac{\sqrt{1 - \sigma^2}}{2}\ln\frac{1 + \sigma}{1 - \sigma}\right) \tag{4.30}$$

where $\sigma^2 = GN/\lambda < 1$. As one sees, $|\mathbf{E}| \to 0$ as $x \to \pm\infty$. The solution obtained satisfies all the regularity conditions and is a solitonian one. The density of mass ρ_m and the density of effective charge ρ_e are

$$\rho_m|_{x\to-\infty} \to \begin{cases} \text{const}, & b = 1 \\ 0, & b > 1 \end{cases}$$

$$\rho_m|_{x\to+\infty} \to 0, \qquad b \geq 1$$

$$\rho_e|_{x\to-\infty} \to \begin{cases} 2C^2\sqrt{G}(1 - \sigma^2)/\pi\sigma, & b = 1 \\ 0, & b > 1 \end{cases}$$

$$\rho_e|_{x\to+\infty} \to 0, \qquad b \geq 1$$

The total charge of the system is equal to zero.

(b) Let us consider the case with $I_c = 0$, choosing

$$P(I) = \lambda I \tag{4.31}$$

In this case we get

$$A(x) = \frac{1}{\sqrt{G}\,\mathrm{sh}(\sqrt{\lambda}Cx)} \tag{4.32}$$

The metric function $e^{2\gamma}$ in this case reads

$$e^{2\gamma} = \mathrm{cth}^2(\sqrt{\lambda}Cx) \tag{4.33}$$

which gives

$$e^{2\gamma}|_{x\to\pm\infty} \to 1, \qquad e^{2\gamma}|_{x\to\pm 0} \to \infty$$

Inasmuch as $e^{2\beta} = e^{-2\gamma+2C_2x}$, $x = x_1 = -\infty$ corresponds to one of the axes of the field configurations. This axis is regular if $\sqrt{\lambda C} > 1$ and $A(x_1) = 0$ and $e^{2\gamma(x_1)} = 1$. So for $e^{2\gamma}|_{x\to\pm 0} \to \infty$, one gets $e^{2\beta}|_{x\to\pm 0} \to 0$, i.e., $x = x_2 = 0$ corresponds to the second, singular axis. In this case the solution obtained is defined on $-\infty \leq x \leq 0$. At $x \to +\infty$, $e^{2\beta} \to \infty$ and $A(x) \to 0$. This means that $x = +\infty$ defines the spatial infinity. In this case the solution is defined on $0 \leq x \leq \infty$ and possesses one singular axis corresponding $x = 0$.

II. Let us now obtain the droplet-like configuration. Choosing $P(I)$ in the form (see Fig. 2)

$$P(J) = J^{1-2/\sigma}[(1 - J)^{1/\sigma} - J^{1/\sigma}]^2(1 - J) \tag{4.34}$$

where $J = GI$, $\sigma = 2n + 1$, $n = 1, 2, 3, \ldots$, one can find an expression for $A(x)$ which is similar to the one in spherically symmetrical case (see Fig. 1):

$$A(x) = (1/\sqrt{G})\left\{1 - \exp\left[-\frac{2C\sqrt{G}}{\sigma}(x - x_0)\right]\right\}^{\sigma/2} \tag{4.35}$$

As one can readily see from (4.35), the conditions (4.22) and (4.23) are fulfilled if $|C_2| \leq C\sqrt{G}/\sigma$. It is noteworthy that at $x \leq x_0$, $A(x) \equiv 0$ and the space-time is flat, the gravitational field being absent (Rybakov *et al.*, 1993).

There is a significant difference between solutions (3.37) and (4.35). For the case of spherical symmetry the droplet-like solution can be transformed to the soliton-like one if the boundary ξ_0 is removed by putting $\xi_0 = 0$ (as in this case $\exp[\beta(\xi_0)] = 1/\xi_0 = \infty$). On the contrary, for the case of cylindrical symmetry the removal of the boundary is equivalent to putting $x_0 = -\infty$, as in this case $\exp[\beta(x_0)] = \exp(-|C_2|x_0) = \infty$. Under this last choice the solution (4.35) takes constant value $A(x) = 1/\sqrt{G}$ and the soliton structure disappears. For the considered case, as well as for that of spherical symmetry, the density of the field energy is given by equation (3.23) and the linear density of energy is similar to (3.24):

$$E_f = (C/4) \int_0^{1/\sqrt{G}} dA\, e^{-3\gamma}[\sqrt{P}(1 + e^{2\gamma}) + 4I(\sqrt{P})_I] \tag{4.36}$$

Substituting $P(I)$ from (4.34) into (4.36), one can find that E_f is finite and the total energy $E_f + E_g$ turns out to be zero.

Let us now define the effective charge density ρ_e and total charge Q corresponding to the unit length on the z axis. In general from (2.4) one gets (Shikin, 1995)

$$j^\alpha = \frac{1}{4\pi} (\varphi_{,\beta}\varphi^{,\beta})\Psi_I A^\alpha \tag{4.37}$$

which for a static radial electric field leads to

$$j^0 = \frac{C^2}{4\pi} e^{-2(\alpha+\gamma)} P_I A \tag{4.38}$$

Then for a chronometric invariant electric charge density ρ_e we have

$$\rho_e = \frac{j^0}{\sqrt{g^{00}}} = \frac{C^2}{4\pi} e^{-(2\alpha+\gamma)} P_I A \tag{4.39}$$

The total charge is defined from the equality

$$Q = 2\pi \int_{x_a}^{x_\infty} \rho_e \sqrt{-^3g}\, dx \tag{4.40}$$

Putting the corresponding quantities into the foregoing equality, we obtain after some simple calculations

$$Q = \tfrac{1}{2} e^{-2\gamma} A' |_{x_a}^{x_\infty} = 0 \tag{4.41}$$

Now it is worthwhile to make again the comparison with the flat-space solutions of equations (2.3) and (2.4), using the interval

$$ds^2 = dt^2 - d\rho^2 - \rho^2\, d\phi^2 - dz^2$$

In this case the scalar field equation (2.3) admits the solution

$$\varphi'(\rho) = CP(I)/\rho, \qquad P(I) = 1/\Psi(I), \qquad C = \text{const} \tag{4.42}$$

Inserting (4.42) into (2.4), one can find the electromagnetic field equation, which admits solution in quadratures:

$$\int \frac{dA}{\sqrt{P(I)}} = \pm C \ln \frac{\rho}{\rho_0}, \qquad \rho_0 = \text{const} \tag{4.43}$$

Substituting $P(I)$ from (4.34) in (4.43), one gets the solution of the droplet-like form:

$$A(\rho) = (1/\sqrt{\lambda})[1 - (\rho/\rho_0)^{2C\sqrt{\lambda}/\sigma}]^{\sigma/2} \tag{4.44}$$

One concludes from (4.44) that $A(\rho \geq \rho_0) \equiv 0$. This means that the electric charge of the system is zero. For the solution (4.34) the regularity conditions both on the axis $\rho = 0$ and on the surface of the critical cylinder $\rho = \rho_0$ are fulfilled if $C\sqrt{\lambda} \geq \sigma$. It is noteworthy that in the case of cylindrical symmetry, both in flat space-time and with account of the proper gravitational field, there do not exist any soliton-like solutions, as for the choice $\rho_0 = \infty$ the solution (4.44) degenerates into a constant: $A(\rho) = 1/\sqrt{\lambda}$. The linear density of the field energy in flat space-time can be found from an expression similar to (3.23), and, as in the case of spherical symmetry, it is equal to zero:

$$E_f = \frac{C}{2} \sqrt{PI}\,\Big|_{A(\rho_0)}^{A(0)} = 0$$

as expected.

5. DISCUSSION

Exact regular static spherically and/or cylindrically symmetrical particle-like solutions to the equations of scalar nonlinear electrodynamics in general relativity have been obtained. As a particular case, we found a class of regular solutions with sharp boundary (droplet-like solutions or simply droplets). It was shown that outside the droplet, gravitational and electromagnetic fields remain absent, i.e., the total energy and total charge of the configuration are zero. We underline once more the significant difference between the droplet-like solutions with spherical symmetry and those with cylindrical symmetry. In the first case there exists the possibility of continuous transformation of the droplet-like configuration into the solitonian one by transporting the sharp boundary to infinity. There is no such possibility for the second case, and the soliton-like configuration disappears when the boundary is smoothed tending to infinity. We intend to study further the interaction processes of droplets with external electromagnetic and gravitational fields and also the scattering of photons and electrons on droplets.

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